

RANK ONE CONNECTIONS ON ABELIAN VARIETIES

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ABSTRACT. Let A be a complex abelian variety. The moduli space \mathcal{M}_C of rank one algebraic connections on A is a principal bundle over the dual abelian variety $A^\vee = \text{Pic}^0(A)$ for the group $H^0(A, \Omega_A^1)$. Take any line bundle L on A^\vee ; let $\mathcal{C}(L)$ be the algebraic principal $H^0(A^\vee, \Omega_{A^\vee}^1)$ -bundle over A^\vee given by the sheaf of connections on L . The line bundle L produces a homomorphism $H^0(A, \Omega_A^1) \rightarrow H^0(A^\vee, \Omega_{A^\vee}^1)$. We prove that $\mathcal{C}(L)$ is isomorphic to the principal $H^0(A^\vee, \Omega_{A^\vee}^1)$ -bundle obtained by extending the structure group of the principal $H^0(A, \Omega_A^1)$ -bundle \mathcal{M}_C using this homomorphism given by L . We compute the ring of algebraic functions on $\mathcal{C}(L)$. As an application of the above result, we show that \mathcal{M}_C does not admit any non-constant algebraic function, despite the fact that it is biholomorphic to $(\mathbb{C}^*)^{2 \dim A}$ implying that it has many non-constant holomorphic functions.

1. INTRODUCTION

Let A be a complex abelian variety of dimension d_0 . Let \mathcal{M} be the moduli space of rank one λ -connections on A (the definition of λ -connections is recalled in Section 2). This \mathcal{M} is a vector bundle over $A^\vee := \text{Pic}^0(A)$, and it fits in the following short exact sequence of vector bundles on A^\vee

$$(1.1) \quad 0 \rightarrow A^\vee \times H^0(A, \Omega_A^1) \rightarrow \mathcal{M} \rightarrow \mathcal{O}_{A^\vee} \rightarrow 0.$$

Take any line bundle L on A^\vee . It produces a homomorphism

$$(1.2) \quad \phi_L^* : H^0(A, \Omega_A^1) \rightarrow H^0(A^\vee, \Omega_{A^\vee}^1);$$

this homomorphism is the pull back of differential forms by the map $A^\vee \rightarrow A = \text{Pic}^0(A^\vee)$ defined by $x \mapsto L^* \otimes \tau_x^* L$, where τ_x is the translation automorphism $z \mapsto z+x$ of A^\vee . Let

$$(1.3) \quad 0 \rightarrow A^\vee \times H^0(A^\vee, \Omega_{A^\vee}^1) = \Omega_{A^\vee}^1 \rightarrow \mathcal{W} \rightarrow \mathcal{O}_{A^\vee} \rightarrow 0$$

be the push-forward of the exact sequence in (1.1) using the homomorphism ϕ_L^* in (1.2).

Let

$$0 \rightarrow \Omega_{A^\vee}^1 \rightarrow \text{At}(L)^* \rightarrow \mathcal{O}_{A^\vee} \rightarrow 0$$

be the dual of the Atiyah exact sequence for L . We prove that it is related to the one in (1.3) in the following way (see Proposition 2.2):

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Proposition 1.1. *There exists an isomorphism of vector bundles $\eta : \mathcal{W} \rightarrow \text{At}(L)^*$ such that the following diagram is commutative*

$$\begin{array}{ccccccc} 0 & \longrightarrow & \Omega_{A^\vee}^1 & \longrightarrow & \mathcal{W} & \longrightarrow & \mathcal{O}_{A^\vee} \longrightarrow 0 \\ & & \parallel & & \downarrow \eta & & \downarrow -1. \\ 0 & \longrightarrow & \Omega_{A^\vee}^1 & \longrightarrow & \text{At}(L)^* & \longrightarrow & \mathcal{O}_{A^\vee} \longrightarrow 0 \end{array}$$

Let \mathcal{M}_C be the moduli space of rank one algebraic connections on A . It is a principal bundle over A^\vee for the additive group $H^0(A, \Omega_A^1)$. Let $\mathcal{C}(L)$ be the fiber bundle over A^\vee given by the sheaf of algebraic connections on L ; it is in fact a principal $H^0(A, \Omega_{A^\vee}^1)$ -bundle over A^\vee .

We prove the following (see Corollary 2.5):

Corollary 1.2. *The principal $H^0(A, \Omega_{A^\vee}^1)$ -bundle $\mathcal{C}(L)$ over A^\vee is isomorphic to the principal $H^0(A, \Omega_{A^\vee}^1)$ -bundle obtained by extending the structure group of the principal $H^0(A, \Omega_A^1)$ -bundle \mathcal{M}_C using the homomorphism ϕ_L^* in (1.2).*

Let V_L^* be the cokernel of the homomorphism ϕ_L^* . There is a projection

$$\rho : \mathcal{C}(L) \rightarrow V_L^*.$$

We prove the following theorem (see Theorem 3.4):

Theorem 1.3. *All algebraic functions on $\mathcal{C}(L)$ factor through the projection ρ . In particular, if ϕ_L^* is an isomorphism, then there are no non-constant algebraic functions on $\mathcal{C}(L)$.*

We note that if either L is ample or L^* is ample, then ϕ_L^* is an isomorphism.

Using Theorem 1.3 it can be shown that there are no non-constant algebraic functions on the moduli space \mathcal{M}_C . (See Proposition 4.1.)

We note that \mathcal{M}_C is canonically biholomorphic to $\text{Hom}(\pi_1(A), \mathbb{C}^*)$; the biholomorphism sends an integrable connection on A to its monodromy. Therefore, \mathcal{M}_C is biholomorphic to $(\mathbb{C}^*)^{2d_0}$. Consequently, \mathcal{M}_C admits non-constant holomorphic functions.

Proposition 4.1 yields the following theorem on the moduli space of rank one λ -connections (see Theorem 4.2):

Theorem 1.4. *All algebraic functions on \mathcal{M} factor through the forgetful map*

$$\mathcal{M} \rightarrow \mathbb{C}$$

that sends any λ -connection (L', c', D') to c' .

2. MODULI SPACE OF λ -CONNECTIONS

Let A be an abelian variety, defined over \mathbb{C} , of dimension d_0 , with $d_0 \geq 1$. The identity element of A will be denoted by 0. The dual abelian variety $\text{Pic}^0(A)$ will be denoted by A^\vee .

Consider a triple (L, c, D) , where L is an algebraic line bundle over A , $c \in \mathbb{C}$, and

$$D : L \longrightarrow L \otimes \Omega_A^1$$

is an algebraic differential operator, of order at most one, satisfying the identity

$$D(f_0 s) = f_0 \cdot D(s) + cs \otimes df_0,$$

where f_0 is a locally defined algebraic function, and s is a locally defined algebraic section of L . It is easy to see that the operator D is integrable, meaning the composition

$$D \circ D : L \longrightarrow L \otimes \Omega_A^2$$

vanishes identically. If $c = 0$, then D is an algebraic one-form on A , so $(L, 0, D)$ is a Higgs line bundle. If $c \neq 0$, then D/c is an integrable algebraic connection on L . Therefore, if $c \neq 0$, then $L \in \text{Pic}^0(A) = A^\vee$.

A λ -connection on A is a triple (L, c, D) of the above form such that $L \in A^\vee$. As mentioned above, if $c \neq 0$, then the condition on L is automatically satisfied.

If (L, c_1, D_1) and (L, c_2, D_2) are two λ -connections, then $(L, c_1 + c_2, D_1 + D_2)$ is also a λ -connection. Similarly, for any $\alpha \in \mathbb{C}$, the triple $(L, \alpha \cdot c_1, \alpha \cdot D_1)$ is a λ -connection. Therefore, the space of all λ -connections on a fixed line bundle is a complex vector space. The dimension of this vector space is $d_0 + 1$.

Let (L, c, D) be a λ -connection on A . For any $\omega \in H^0(A, \Omega_A^1)$, the triple $(L, c, D + \omega)$ is also a λ -connection. Moreover, if (L, c, D') is another λ -connection, then there is a unique $\omega' \in H^0(A, \Omega_A^1)$ such that $D' = D + \omega'$. In other words, for fixed L and c , the space of all λ -connections of the form (L, c, D'') is an affine space for $H^0(A, \Omega_A^1)$.

Let \mathcal{M} be the moduli space of λ -connections on A . The moduli space of λ -connections of arbitrary but fixed rank was constructed in [Si]. Here we are considering only λ -connections of rank one. It is substantially simpler to construct the moduli space of λ -connections of rank one.

Let

$$(2.1) \quad f : \mathcal{M} \longrightarrow A^\vee$$

be the projection defined by $(L, c, D) \longmapsto L$. It was noted above that each fiber of f is a vector space of dimension $d_0 + 1$. The projection f makes \mathcal{M} the total space of an algebraic vector bundle over A^\vee of rank $d_0 + 1$. This vector bundle will also be denoted by \mathcal{M} . Let

$$(2.2) \quad q_0 : \mathcal{M} \longrightarrow A^\vee \times \mathbb{C}$$

be the projection defined by $(L, c, D) \longmapsto (L, c)$. Let

$$(2.3) \quad p_{\mathbb{C}} : A^\vee \times \mathbb{C} \longrightarrow \mathbb{C}$$

be the natural projection. The restriction of the morphism $p_{\mathbb{C}} \circ q_0$ to a fiber of f is linear projection (recall that the fibers of f are vector spaces). The kernel of this linear projection is $H^0(A, \Omega_A^1)$. Therefore, we get a short exact sequence of vector bundles on A^\vee

$$(2.4) \quad 0 \longrightarrow A^\vee \times H^0(A, \Omega_A^1) \xrightarrow{\iota} \mathcal{M} \xrightarrow{q_0} \mathcal{O}_{A^\vee} = A^\vee \times \mathbb{C} \longrightarrow 0,$$

where $A^\vee \times H^0(A, \Omega_A^1)$ is the trivial vector bundle over A^\vee with fiber $H^0(A, \Omega_A^1)$. This trivial vector bundle $A^\vee \times H^0(A, \Omega_A^1)$ will be denoted by \mathcal{V}_0 .

Let

$$(2.5) \quad \delta \in H^1(A^\vee, \mathcal{V}_0) = H^1(A^\vee, \mathcal{O}_{A^\vee}) \otimes H^0(A, \Omega_A^1) = H^1(A^\vee, \mathcal{O}_{A^\vee}) \otimes H^0(A, TA)^*$$

be the extension class for the short exact sequence in (2.4).

There is a canonical isomorphism

$$(2.6) \quad H^0(A, TA) \longrightarrow H^1(A^\vee, \mathcal{O}_{A^\vee})$$

(see [Mu, p. 4, Theorem]); note that since $A = (A^\vee)^\vee$, and the homomorphism of evaluation of sections at the origin $H^0(A, TA) \longrightarrow T_0A$ is an isomorphism, the isomorphism in (2.6) follows from the above mentioned result of [Mu]. (We will recall a construction of the isomorphism (2.6) in the proof of Lemma 2.1.) The isomorphism in (2.6) gives an element

$$(2.7) \quad c_0 \in H^1(A^\vee, \mathcal{O}_{A^\vee}) \otimes H^0(A, TA)^*.$$

Lemma 2.1. *The cohomology class c_0 in (2.7) coincides with $-\delta$, where δ is the class in (2.5).*

Proof. Express A as V/Λ , where V is a complex vector space of dimension d_0 , and Λ is a cocompact lattice in V . Let $\bar{\Lambda}$ be the lattice in \bar{V}^* given by Λ . This lattice $\bar{\Lambda}$ is defined by the condition that the natural pairing $\Lambda \times \bar{\Lambda} \longrightarrow \mathbb{C}$ takes values in \mathbb{Z} . The abelian variety A^\vee is the quotient $\bar{V}^*/\bar{\Lambda}$.

We note that $H^0(A, TA) = V$, because $H^0(A, TA) = T_0A = T_0V = V$ (the fiber T_0A is identified with $H^0(A, TA)$ by evaluating vector fields at 0). So, $H^0(A, \Omega_A^1) = V^*$. The conjugate-linear homomorphism

$$H^0(A, \Omega_A^1) \longrightarrow H^1(A, \mathcal{O}_A)$$

defined by $\theta \longmapsto \bar{\theta}$ is an isomorphism (the closed form $\bar{\theta}$ represents a class in the Dolbeault cohomology). Hence

$$(2.8) \quad H^1(A, \mathcal{O}_A) = \bar{V}^*.$$

Consequently,

$$(2.9) \quad H^1(A^\vee, \mathcal{O}_{A^\vee}) = \overline{\bar{V}^*}^* = V.$$

The isomorphism in (2.6) is the identity map of V .

A line bundle L over A with $c_1(L) = 0$ admits a unique unitary flat connection. This unitary flat connection on L will be denoted by ∇^L . So we get a C^∞ section of the vector bundle \mathcal{M}

$$(2.10) \quad s : A^\vee \longrightarrow \mathcal{M}$$

defined by $L \longmapsto (L, 1, \nabla^L)$. Note that the map

$$\mathcal{O}_{A^\vee} \longrightarrow \mathcal{M}$$

defined by $(L, c) \longmapsto c \cdot s(L)$ gives a C^∞ splitting of the short exact sequence in (2.4).

Let $\bar{\partial}_{\mathcal{M}} : C^\infty(A^\vee, \mathcal{M}) \rightarrow C^\infty(A^\vee, \mathcal{M} \otimes (T^{0,1}A^\vee)^*)$ be the Dolbeault operator for the holomorphic vector bundle \mathcal{M} . Since q_0 in (2.4) is holomorphic,

$$(q_0 \times \text{Id})(\bar{\partial}_{\mathcal{M}}(s)) = \bar{\partial}(q_0(s)) = \bar{\partial}(1) = 0.$$

Consequently, $\bar{\partial}_{\mathcal{M}}(s)$ is a $(0,1)$ -form on A^\vee with values in the vector bundle $\mathcal{V}_0 = A^\vee \times H^0(A, \Omega_A^1)$. The extension class δ in (2.5) is represented by the \mathcal{V}_0 -valued $(0,1)$ -form $\bar{\partial}_{\mathcal{M}}(s)$.

Consider the trivial C^∞ line bundle $A \times \mathbb{C}$ over A . The $(0,1)$ -component of the de Rham differential, namely $\bar{\partial}$, defines the trivial holomorphic structure on it. Any other Dolbeault operator on this C^∞ line bundle is of the form $\bar{\partial} + \omega'$, where ω' is a $(0,1)$ -form with $\bar{\partial}\omega' = 0$. For any $\omega \in H^0(A, \Omega_A^1)$, the $(0,1)$ -form $\omega' := \bar{\omega}$ satisfies this condition. Therefore, we have a map

$$(2.11) \quad \psi : \overline{H^0(A, \Omega_A^1)} \rightarrow A^\vee$$

that sends any $\omega \in H^0(A, \Omega_A^1)$ to the holomorphic line bundle defined by the Dolbeault operation $\bar{\partial} + \bar{\omega}$ on the trivial line bundle. This ψ is a holomorphic covering map.

Consider the section s constructed in (2.10) and the map ψ in (2.11). For any $\omega \in H^0(A, \Omega_A^1)$, the holomorphic connection $s(\psi(\omega))$ is $\bar{\partial} - \omega$ on the topologically trivial line bundle $A \times \mathbb{C}$, where $\bar{\partial}$ is the $(1,0)$ -component of the de Rham differential. (The corresponding unitary connection is $d + \bar{\omega} - \omega$.) Since the extension class δ in (2.5) is represented by $\bar{\partial}_{\mathcal{M}}(s)$, we conclude that δ coincides with $-c_0$. \square

Take any algebraic line bundle L over A^\vee . Using L , we get a homomorphism

$$(2.12) \quad \phi_L : H^0(A^\vee, TA^\vee) \rightarrow H^0(A, TA);$$

we will recall the construction of ϕ_L . Consider

$$c_1(L) \in H^{1,1}(A^\vee) = H^1(A^\vee, \Omega_{A^\vee}^1).$$

Using the obvious pairing of TA^\vee with its dual $\Omega_{A^\vee}^1$, we get a homomorphism

$$H^{1,1}(A^\vee) \otimes H^0(A^\vee, TA^\vee) \rightarrow H^1(A^\vee, \mathcal{O}_{A^\vee}).$$

Using it, $c_1(L)$ gives a homomorphism $H^0(A^\vee, TA^\vee) \rightarrow H^1(A^\vee, \mathcal{O}_{A^\vee})$. The homomorphism ϕ_L in (2.12) coincides with this homomorphism using the identification

$$H^1(A^\vee, \mathcal{O}_{A^\vee}) = H^0(A, TA)$$

(see (2.9)).

Another construction of ϕ_L is as follows. For any $x \in A^\vee$, let τ_x be the translation automorphism of A^\vee defined by $z \mapsto x + z$. The map

$$\tilde{\phi}_L : A^\vee \rightarrow (A^\vee)^\vee = A$$

defined by $x \mapsto (\tau_x^* L) \otimes L^*$ is a homomorphism of algebraic groups (see [Mu, pp. 59–60, Corollary 4]). The homomorphism ϕ_L in (2.12) is the corresponding homomorphism of Lie algebras.

Fix a line bundle L on A^\vee . Let

$$(2.13) \quad 0 \rightarrow \mathcal{O}_{A^\vee} \rightarrow \text{At}(L) \rightarrow TA^\vee \rightarrow 0$$

be the Atiyah exact sequence for L ; see [At, p. 187, Theorem 1] for the construction of Atiyah exact sequence. Let

$$(2.14) \quad 0 \longrightarrow \Omega_{A^\vee}^1 \longrightarrow \text{At}(L)^* \xrightarrow{p_1} \mathcal{O}_{A^\vee} \longrightarrow 0$$

be the dual of the exact sequence in (2.13).

Let

$$(2.15) \quad \phi_L^* : H^0(A, \Omega_A^1) \longrightarrow H^0(A^\vee, \Omega_{A^\vee}^1)$$

be the dual of the homomorphism ϕ_L in (2.12). Using ϕ_L^* , we will construct a short exact sequence from the one in (2.4).

Consider the direct sum of vector bundles $(A^\vee \times H^0(A^\vee, \Omega_{A^\vee}^1)) \oplus \mathcal{M}$ over A^\vee , where $A^\vee \times H^0(A^\vee, \Omega_{A^\vee}^1)$ is the trivial vector bundle with fiber $H^0(A^\vee, \Omega_{A^\vee}^1)$. Let

$$A^\vee \times H^0(A, \Omega_A^1) \longrightarrow (A^\vee \times H^0(A^\vee, \Omega_{A^\vee}^1)) \oplus \mathcal{M}$$

be the embedding defined by $(x, v) \mapsto ((x, -\phi_L^*(v)), \iota(x, v))$, where ι is the homomorphism in (2.4) and ϕ_L^* is constructed in (2.15). Let

$$(2.16) \quad \mathcal{W} := ((A^\vee \times H^0(A^\vee, \Omega_{A^\vee}^1)) \oplus \mathcal{M}) / (A^\vee \times H^0(A, \Omega_A^1))$$

be the quotient of this injective homomorphism. Note that \mathcal{W} has a projection to \mathcal{O}_{A^\vee} defined by $(a, b) \mapsto q_0(b)$, where q_0 is the projection in (2.4), and $a \in A^\vee \times H^0(A^\vee, \Omega_{A^\vee}^1)$. Using this projection, the vector bundle \mathcal{W} fits in a short exact sequence

$$(2.17) \quad 0 \longrightarrow A^\vee \times H^0(A^\vee, \Omega_{A^\vee}^1) = \Omega_{A^\vee}^1 \longrightarrow \mathcal{W} \xrightarrow{p_2} \mathcal{O}_{A^\vee} \longrightarrow 0.$$

Proposition 2.2. *There exists an isomorphism of vector bundles $\eta : \mathcal{W} \longrightarrow \text{At}(L)^*$ such that the following diagram is commutative*

$$\begin{array}{ccccccc} 0 & \longrightarrow & \Omega_{A^\vee}^1 & \longrightarrow & \mathcal{W} & \longrightarrow & \mathcal{O}_{A^\vee} \longrightarrow 0 \\ & & \parallel & & \downarrow \eta & & \downarrow -1. \\ 0 & \longrightarrow & \Omega_{A^\vee}^1 & \longrightarrow & \text{At}(L)^* & \longrightarrow & \mathcal{O}_{A^\vee} \longrightarrow 0 \end{array}$$

where and the top and bottom exact sequences are those in (2.17) and (2.14) respectively.

Proof. The extension class in $H^1(A^\vee, \Omega_{A^\vee}^1)$ for the Atiyah exact sequence for L in (2.13) coincides with $c_1(L)$ [At, p. 196, Proposition 12]. Therefore, the extension class for the dual exact sequence in (2.14) is also $c_1(L)$.

Using the homomorphism ϕ_L^* in (2.15), the cohomology class

$$c_0 \in H^1(A^\vee, \mathcal{O}_{A^\vee}) \otimes H^0(A, \Omega_A^1)$$

in (2.7) produces a cohomology class

$$(2.18) \quad \tilde{c}_0 \in H^1(A^\vee, \mathcal{O}_{A^\vee}) \otimes H^0(A^\vee, \Omega_{A^\vee}^1) = H^1(A^\vee, \Omega_{A^\vee}^1).$$

Similarly, using ϕ_L^* , the cohomology class $\delta \in H^1(A^\vee, \mathcal{O}_{A^\vee}) \otimes H^0(A, \Omega_A^1)$ in (2.5) produces a cohomology class

$$(2.19) \quad \tilde{\delta}_0 \in H^1(A^\vee, \Omega_{A^\vee}^1).$$

From the constructions of the vector bundle \mathcal{W} and the sequence in (2.17) it follows immediately that the extension class for the short exact sequence in (2.17) coincides with $\tilde{\delta}_0$.

From Lemma 2.1 it follows that $-\tilde{c}_0$ in (2.18) coincides with $\tilde{\delta}_0$ in (2.19). Therefore, the extension class for the short exact sequence in (2.17) is $-\tilde{c}_0$.

The cohomology class \tilde{c}_0 coincides with $c_1(L)$; this follows from the expression of $c_1(L)$ in [Mu, p. 18, Proposition] and the computation in [Mu, p. 83]. Since the extension classes for the short exact sequences in (2.17) and (2.14) are $-\tilde{c}_0$ and \tilde{c}_0 respectively, the proof of the proposition is complete. \square

Remark 2.3. It should be emphasized that Proposition 2.2 does not imply that the isomorphism class of the short exact sequence in (2.14) is independent of L , because the homomorphism ϕ_L^* depends on L . Note that the short exact sequence in (2.14) splits if and only if L admits a holomorphic connection [At, p. 188, Definition]. Hence the short exact sequence in (2.14) splits if L is topologically trivial, and it does not split if L is ample.

Consider the projections p_1 and p_2 in (2.14) and (2.17) respectively. Note that

$$p_1^{-1}(A^\vee \times \{1\}) \quad \text{and} \quad p_2^{-1}(A^\vee \times \{1\})$$

are principal bundles over A^\vee for the additive group $H^0(A^\vee, \Omega_{A^\vee}^1)$.

We have the following corollary of Proposition 2.2.

Corollary 2.4. *The two principal $H^0(A^\vee, \Omega_{A^\vee}^1)$ -bundles $p_1^{-1}(A^\vee \times \{1\})$ and $p_2^{-1}(A^\vee \times \{1\})$ are isomorphic.*

Proof. Consider the isomorphism η in Proposition 2.2. The restriction of $-\eta$ to $p_2^{-1}(A^\vee \times \{1\})$ is the required isomorphism. \square

Consider $q_0^{-1}(A^\vee \times \{1\})$, where q_0 is the projection in (2.4). It is a principal bundle over A^\vee for the additive group $H^0(A, \Omega_A^1)$. From the constructions of the vector bundle \mathcal{W} and exact sequence in (2.17) it follows that the principal $H^0(A^\vee, \Omega_{A^\vee}^1)$ -bundle $p_2^{-1}(A^\vee \times \{1\})$ over A^\vee is identified with the one obtained by extending the structure group of the principal $H^0(A, \Omega_A^1)$ -bundle $q_0^{-1}(A^\vee \times \{1\})$ using the homomorphism ϕ_L^* in (2.15).

Therefore, the following is a reformulation of Corollary 2.4.

Corollary 2.5. *The principal $H^0(A^\vee, \Omega_{A^\vee}^1)$ -bundle $p_1^{-1}(A^\vee \times \{1\})$ is isomorphic to the one obtained by extending the structure group of the principal $H^0(A, \Omega_A^1)$ -bundle $q_0^{-1}(A^\vee \times \{1\})$ using the homomorphism ϕ_L^* .*

3. ALGEBRAIC FUNCTIONS ON SHEAF OF CONNECTIONS

Let A be a complex abelian variety, with $\dim A > 0$. Let ξ be an algebraic line bundle over A . Let

$$(3.1) \quad 0 \longrightarrow \Omega_A^1 \xrightarrow{\iota_1} \text{At}(\xi)^* \xrightarrow{q} \mathcal{O}_A \longrightarrow 0$$

be the dual of the Atiyah exact sequence for ξ . Let

$$(3.2) \quad q_1 : \text{At}(\xi)^* \longrightarrow \mathbb{C}$$

be the composition of q and the projection from the total space of \mathcal{O}_A to \mathbb{C} . The sheaf of sections of $\text{At}(\xi)^*$ is the sheaf of λ -connections on the line bundle ξ . Define the fiber bundle over A

$$(3.3) \quad \mathcal{C}(\xi) := q_1^{-1}(1) \subset \text{At}(\xi)^*.$$

The sheaf of holomorphic (respectively, algebraic) sections of $\mathcal{C}(\xi)$ is the sheaf of holomorphic (respectively, algebraic) connections on ξ . Using the homomorphism ι_1 in (3.1), the fiber bundle $\mathcal{C}(\xi)$ is a principal bundle over A for the group $H^0(A, \Omega_A^1)$.

Let

$$(3.4) \quad \mathbb{P} := P(\text{At}(\xi)^*) \quad \text{and} \quad \mathbb{P}_0 := P(\Omega_A^1)$$

be the projective bundles over A parametrizing lines in $\text{At}(\xi)^*$ and Ω_A^1 respectively. Using ι_1 in (3.1), the projective bundle \mathbb{P}_0 is a subbundle of \mathbb{P} , and

$$(3.5) \quad \mathcal{C}(\xi) = \mathbb{P} \setminus \mathbb{P}_0.$$

Let $\mathcal{O}_{\mathbb{P}}(-1) \longrightarrow \mathbb{P}$ be the tautological line bundle. Let

$$(3.6) \quad \sigma \in H^0(\mathbb{P}, \mathcal{O}_{\mathbb{P}}(1))$$

be the section of $\mathcal{O}_{\mathbb{P}}(1) := \mathcal{O}_{\mathbb{P}}(-1)^*$ defined by $\sigma(v) = q_1(v) \in \mathbb{C}$, where $v \in \mathcal{O}_{\mathbb{P}}(-1)$, and q_1 is the projection in (3.2). The divisor $\text{Div}(\sigma)$ coincides with \mathbb{P}_0 in (3.4).

Let $\mathbb{C}[\mathcal{C}(\xi)]$ be the space of algebraic functions on the variety $\mathcal{C}(\xi)$. From (3.5) it follows that $\mathbb{C}[\mathcal{C}(\xi)]$ is identified with the direct limit

$$\lim_{n \rightarrow \infty} H^0(\mathbb{P}, \mathcal{O}_{\mathbb{P}}(n\mathbb{P}_0)).$$

Since $\text{Div}(\sigma) = \mathbb{P}_0$,

$$(3.7) \quad \mathbb{C}[\mathcal{C}(\xi)] = \lim_{n \rightarrow \infty} H^0(\mathbb{P}, \mathcal{O}_{\mathbb{P}}(n)),$$

where $\mathcal{O}_{\mathbb{P}}(n) = \mathcal{O}_{\mathbb{P}}(1)^{\otimes n}$; the vector space $H^0(\mathbb{P}, \mathcal{O}_{\mathbb{P}}(n))$ sits inside $H^0(\mathbb{P}, \mathcal{O}_{\mathbb{P}}(n+1))$ using the homomorphism $s \longmapsto s \otimes \sigma$.

Our aim in this section is to compute the algebra $\mathbb{C}[\mathcal{C}(\xi)]$.

For any $x \in A$, let $\tau_x : A \longrightarrow A$ be the automorphism defined by $z \longmapsto z + x$. Let

$$A_0 \subset A$$

be the closed subgroup defined by all x such that $\tau_x^* \xi = \xi$ (see [Mu, pp. 59–60, Corollary 4]). Let

$$(3.8) \quad V_{\xi} := \text{Lie}(A_0) = T_0 A_0 \subset T_0 A \subset H^0(A, TA)$$

be the tangent space, where $0 \in A$ is the identity element. The subspace V_{ξ} in (3.8) defines an algebraic foliation on A . This foliation will be denoted by \mathcal{F} , so $\mathcal{F} = A \times V_{\xi}$. The evaluation of sections of \mathcal{F} at 0 gives an isomorphism

$$H^0(A, \mathcal{F}) \xrightarrow{\sim} V_{\xi}.$$

Take any Zariski open subset $U \subset A$. A *partial connection* on $\xi_U := \xi|_U$ in the direction of \mathcal{F} is an algebraic first order differential operator

$$D : \xi_U \longrightarrow \xi_U \otimes \mathcal{F}^*$$

satisfying the identity

$$D(f_0 s) = f_0 \cdot D(s) + \widetilde{df}_0 \otimes s,$$

where f_0 is a locally defined function, s is a locally defined section of ξ_U , and \widetilde{df}_0 is the projection of the 1-form df_0 to $\mathcal{F}^*|_U$.

Let $\mathcal{C}_{\mathcal{F}}(\xi)$ be the sheaf of partial connections on ξ in the direction of \mathcal{F} .

We will give an alternative description of $\mathcal{C}_{\mathcal{F}}(\xi)$.

Consider the projection $\iota_1^* : \text{At}(\xi) \longrightarrow TA$, where ι_1 is the homomorphism in (3.1). Define

$$\text{At}_{\mathcal{F}}(\xi) = (\iota_1^*)^{-1}(\mathcal{F}).$$

We have a short exact sequence of vector bundles

$$(3.9) \quad 0 \longrightarrow \mathcal{F}^* \xrightarrow{\iota_1'} \text{At}_{\mathcal{F}}(\xi)^* \xrightarrow{q_1'} \mathcal{O}_A \longrightarrow 0;$$

it coincides with (3.1) if $\mathcal{F} = TA$. The fiber bundle $\mathcal{C}_{\mathcal{F}}(\xi) \longrightarrow A$ is identified with the inverse image $(q_1')^{-1}(A \times \{1\})$. We note that $\mathcal{C}_{\mathcal{F}}(\xi) = (q_1')^{-1}(A \times \{1\})$ is a principal bundle over A for the additive group $H^0(A, \mathcal{F})$.

We will show that ξ admits a partial connection in the direction of \mathcal{F} .

Let

$$(3.10) \quad \phi_{\xi} : H^0(A, TA) \longrightarrow H^0(A^{\vee}, TA^{\vee})$$

be the homomorphism constructed as in (2.12) using the line bundle ξ .

Lemma 3.1. *The subspace V_{ξ} in (3.8) coincides with the kernel of the homomorphism ϕ_{ξ} in (3.10).*

Proof. This lemma follows immediately by comparing [Mu, p. 18, Proposition] and [Mu, p. 84, Proposition (iii)]. \square

Lemma 3.2. *The short exact sequence in (3.9) splits.*

Proof. We have a commutative diagram

$$(3.11) \quad \begin{array}{ccccccc} 0 & \longrightarrow & \Omega_A^1 & \longrightarrow & \text{At}(\xi)^* & \longrightarrow & \mathcal{O}_A \longrightarrow 0 \\ & & \downarrow & & \downarrow & & \parallel \\ 0 & \longrightarrow & \mathcal{F}^* & \longrightarrow & \text{At}_{\mathcal{F}}(\xi)^* & \longrightarrow & \mathcal{O}_A \longrightarrow 0 \end{array}$$

where the top and the bottom exact sequences are as in (3.1) and (3.9) respectively. Therefore, the extension class for the bottom exact sequence is given by the extension class for the top exact sequence.

We have an isomorphism

$$(3.12) \quad H^1(A, \Omega_A^1) = \text{Hom}_{\mathbb{C}}(H^0(A, TA), H^0(A^{\vee}, TA^{\vee})),$$

because $H^1(A, \mathcal{O}_A) = H^0(A^\vee, TA^\vee)$ (see (2.6)). By this isomorphism, the extension class in $H^1(A, \Omega_A^1)$ for the short exact sequence in (3.1) is mapped to the homomorphism ϕ_ξ in (3.10) (see the proof of Proposition 2.2). We also have

$$\begin{aligned} H^1(A, \mathcal{F}^*) &= H^0(A, \mathcal{F}^*) \otimes H^1(A, \mathcal{O}_A) = V_\xi^* \otimes H^0(A^\vee, TA^\vee) \\ &= \text{Hom}_{\mathbb{C}}(V_\xi, H^0(A^\vee, TA^\vee)), \end{aligned}$$

where V_ξ is defined in (3.8). Combining this with the isomorphism in (3.12) we get an isomorphism

$$\begin{aligned} &\text{Hom}_{\mathbb{C}}(H^1(A, \Omega_A^1), H^1(A, \mathcal{F}^*)) \\ &= \text{Hom}_{\mathbb{C}}(\text{Hom}_{\mathbb{C}}(H^0(A, TA), H^0(A^\vee, TA^\vee)), \text{Hom}_{\mathbb{C}}(V_\xi, H^0(A^\vee, TA^\vee))). \end{aligned}$$

This isomorphism takes the quotient map

$$H^1(A, \Omega_A^1) \longrightarrow H^1(A, \mathcal{F}^*)$$

to the natural quotient map

$$\text{Hom}_{\mathbb{C}}(H^0(A, TA), H^0(A^\vee, A^\vee)) \longrightarrow \text{Hom}_{\mathbb{C}}(V_\xi, H^0(A^\vee, A^\vee))$$

defined by the restriction of homomorphisms to the subspace V_ξ .

The extension class for the top exact sequence in (3.11) is $c_1(\xi)$ [At, p. 196, Proposition 12]. The homomorphism ϕ_ξ in (3.10) is sent to $c_1(\xi)$ by the isomorphism in (3.12) (see the proof of Proposition 2.2).

In view of the above description of the quotient map

$$H^1(A, \Omega_A^1) \longrightarrow H^1(A, \mathcal{F}^*),$$

from Lemma 3.1 we now conclude that the extension class in $H^1(A, \mathcal{F}^*)$ for the short exact sequence in (3.9) vanishes. This completes the proof of the lemma. \square

Fix a splitting homomorphism

$$(3.13) \quad \gamma : \text{At}_{\mathcal{F}}(\xi)^* \longrightarrow V_\xi^*$$

for the short exact sequence in (3.9) (recall that $\mathcal{F} = A \times V_\xi$); so $\gamma \circ \iota'_1(x, v) = v$, where $x \in A$, $v \in V_\xi^*$, and ι'_1 is the homomorphism in (3.9).

Consider the dual homomorphism $\text{At}(\xi)^* \longrightarrow \text{At}_{\mathcal{F}}(\xi)^*$ of the natural inclusion map $\text{At}_{\mathcal{F}}(\xi) \hookrightarrow \text{At}(\xi)$. The restriction of it to $\mathcal{C}(\xi)$ (defined in (3.3)) coincides with the map that sends a locally defined connection on ξ to the partial connection given by it. Let

$$(3.14) \quad \rho : \mathcal{C}(\xi) \longrightarrow V_\xi^*$$

be the surjective map obtained by composing this restriction map with γ in (3.13).

For any nonnegative integer d , let \mathcal{D}_d be the space of algebraic functions on V_ξ^* satisfying the condition that the order of pole at infinity is at most d . So,

$$(3.15) \quad \mathcal{D}_d = \text{Sym}^d(V_\xi \oplus \mathbb{C}).$$

In view of (3.15), the pull back of functions on V_ξ^* by the map ρ in (3.14) yields an inclusion

$$(3.16) \quad \text{Sym}^d(V_\xi \oplus \mathbb{C}) \subset H^0(\mathbb{P}, \mathcal{O}_{\mathbb{P}}(d))$$

(see (3.7) for $H^0(\mathbb{P}, \mathcal{O}_{\mathbb{P}}(d))$).

Proposition 3.3. *For all $d \geq 1$,*

$$\dim H^0(\mathbb{P}, \mathcal{O}_{\mathbb{P}}(d)) \leq \dim \operatorname{Sym}^d(V_{\xi} \oplus \mathbb{C}).$$

Proof. We first note that $H^0(\mathbb{P}, \mathcal{O}_{\mathbb{P}}(d)) = H^0(A, \operatorname{Sym}^d(\operatorname{At}(\xi)))$. Taking symmetric power of the vector bundles in the Atiyah exact sequence

$$(3.17) \quad 0 \longrightarrow \mathcal{O}_A \longrightarrow \operatorname{At}(\xi) \longrightarrow TA \longrightarrow 0,$$

we have a short exact sequence of vector bundles

$$(3.18) \quad 0 \longrightarrow \operatorname{Sym}^n(\operatorname{At}(\xi)) \longrightarrow \operatorname{Sym}^{n+1}(\operatorname{At}(\xi)) \longrightarrow \operatorname{Sym}^{n+1}(TA) \longrightarrow 0$$

for each $n \geq 0$ (by convention, Sym^0 of a nonzero vector space is \mathbb{C}). Let

$$(3.19) \quad \begin{aligned} 0 \longrightarrow H^0(A, \operatorname{Sym}^n(\operatorname{At}(\xi))) &\longrightarrow H^0(A, \operatorname{Sym}^{n+1}(\operatorname{At}(\xi))) \\ &\longrightarrow H^0(A, \operatorname{Sym}^{n+1}(TA)) \xrightarrow{\beta'} H^1(A, \operatorname{Sym}^n(\operatorname{At}(\xi))) \longrightarrow \end{aligned}$$

be the long exact sequence of cohomologies for (3.18).

The projection $\operatorname{Sym}^n(\operatorname{At}(\xi)) \longrightarrow \operatorname{Sym}^n(TA)$ in (3.18) gives a homomorphism

$$H^1(A, \operatorname{Sym}^n(\operatorname{At}(\xi))) \longrightarrow H^1(A, \operatorname{Sym}^n(TA)).$$

Let

$$(3.20) \quad \beta_{n+1} : H^0(A, \operatorname{Sym}^{n+1}(TA)) \longrightarrow H^1(A, \operatorname{Sym}^n(TA))$$

be the composition of this homomorphism and β' in (3.19).

From (3.19) it follows that for all $d \geq 1$,

$$(3.21) \quad \dim H^0(A, \operatorname{Sym}^d(\operatorname{At}(\xi))) \leq 1 + \sum_{i=1}^d \dim \operatorname{kernel}(\beta_i),$$

where β_i is constructed in (3.20).

We will denote the vector space $H^0(A, TA)$ by W . Then,

$$H^0(A, \operatorname{Sym}^{n+1}(TA)) = \operatorname{Sym}^{d+1}(W).$$

Also,

$$H^1(A, \operatorname{Sym}^n(TA)) = H^1(A, \mathcal{O}_A) \otimes H^0(A, \operatorname{Sym}^n(TA)) = \overline{W}^* \otimes \operatorname{Sym}^n(W)$$

(see (2.8)), and

$$H^1(A, \Omega_A^1) = H^1(A, \mathcal{O}_A) \otimes H^0(A, \Omega_A^1) = \overline{W}^* \otimes W^* = \operatorname{Hom}_{\mathbb{C}}(W, \overline{W}^*).$$

The homomorphism

$$\beta_{i+1} : H^0(A, \operatorname{Sym}^{i+1}(TA)) = \operatorname{Sym}^{i+1}(W) \longrightarrow H^1(A, \operatorname{Sym}^i(TA)) = \overline{W}^* \otimes \operatorname{Sym}^i(W)$$

in (3.20) is given by

$$c_1(\xi) \in H^1(A, \Omega_A^1) = \operatorname{Hom}_{\mathbb{C}}(W, \overline{W}^*)$$

using the obvious pairing of W with W^* . Indeed, this follows from the fact that the extension class for the short exact sequence in (3.17) is given by $c_1(\xi)$ using the pairing between W and W^* . In view of Lemma 3.1, we now conclude that

$$(3.22) \quad \text{kernel}(\beta_{i+1}) = \text{Sym}^{i+1}(V_\xi) \subset \text{Sym}^{i+1}(V).$$

The proposition follows from (3.21) and (3.22). \square

Theorem 3.4. *For all $d \geq 1$,*

$$\text{Sym}^d(V_\xi \oplus \mathbb{C}) = H^0(\mathbb{P}, \mathcal{O}_{\mathbb{P}}(d)).$$

All algebraic functions on $\mathcal{C}(\xi)$ factor through the surjective map ρ in (3.14).

Proof. In view of Proposition 3.3, the inclusion homomorphism

$$\text{Sym}^d(V_\xi \oplus \mathbb{C}) \hookrightarrow H^0(\mathbb{P}, \mathcal{O}_{\mathbb{P}}(d))$$

in (3.16) is an isomorphism.

Using (3.7) together with the above isomorphism, it follows that all algebraic functions on $\mathcal{C}(\xi)$ are of the form $g \circ \rho$, where g is an algebraic function on V_ξ^* . \square

4. ALGEBRAIC FUNCTIONS ON THE MODULI SPACE OF λ -CONNECTIONS

In this section we will use the set-up and notation of Section 2.

Let \mathcal{M}_C be the moduli space of algebraic connections of rank one on A . So,

$$(4.1) \quad \mathcal{M}_C = q_0^{-1}(A \times \{1\}) \subset \mathcal{M},$$

where q_0 is defined in (2.2).

Proposition 4.1. *There are no non-constant algebraic functions on \mathcal{M}_C .*

Proof. Fix a line bundle $L \rightarrow A^\vee$ such that the homomorphism ϕ_L in (2.12) is an isomorphism. For example, if L is ample, then ϕ_L is an isomorphism (this follows from the fact that statement (3) in “Theorem of Lefschetz” in [Mu, pp. 29–30] implies statement (2) in the same theorem). Since ϕ_L is an isomorphism, the homomorphism ϕ_L^* in (2.15) is also an isomorphism.

Consider the vector bundle \mathcal{W} constructed in (2.16). Since ϕ_L^* is an isomorphism, the map of fiber bundles

$$\psi : \mathcal{M} \rightarrow \mathcal{W}$$

that sends any $z \in \mathcal{M}$ to the equivalence class of $(0, z) \in (A^\vee \times H^0(A^\vee, \Omega_{A^\vee}^1)) \oplus \mathcal{M}$ is an isomorphism. Therefore,

$$-\eta \circ \psi : \mathcal{M} \rightarrow \text{At}(L^*)$$

is an isomorphism, where η is the isomorphism in Proposition 2.2. Furthermore,

$$p_1 \circ (-\eta \circ \psi) = q_0,$$

where q_0 and p_1 are the homomorphisms in (2.2) and (2.14) respectively. Now from the commutativity of the diagram of maps in Proposition 2.2 we have

$$(4.2) \quad p_1^{-1}(A^\vee \times \{1\}) = q_0^{-1}(A \times \{1\}) = \mathcal{M}_C.$$

Since the homomorphism ϕ_L is an isomorphism, from Theorem 3.4 we know that all algebraic functions on $p_1^{-1}(A^\vee \times \{1\})$ are constant functions. Hence from (4.2) it follows that there are no non-constant functions on \mathcal{M}_C . \square

As mentioned in the introduction, the moduli space \mathcal{M}_C is biholomorphic to

$$\mathrm{Hom}(\pi_1(A), \mathbb{C}^*) \cong (\mathbb{C}^*)^{2d_0}$$

by sending an integrable connection to its monodromy representation. Since the variety $(\mathbb{C}^*)^{2d_0}$ has nonconstant algebraic functions, we conclude that \mathcal{M}_C is not algebraically isomorphic to $(\mathbb{C}^*)^{2d_0}$. (See [Se, p. 101, Remark] for an example of similar phenomenon.)

Theorem 4.2. *All algebraic functions on \mathcal{M} factor through the surjective map*

$$p_{\mathbb{C}} \circ q_0 : \mathcal{M} \longrightarrow \mathbb{C},$$

where q_0 and $p_{\mathbb{C}}$ are defined in (2.2) and (2.3) respectively.

Proof. For any nonzero number $t \in \mathbb{C}$, the inverse image $(p_{\mathbb{C}} \circ q_0)^{-1}(t)$ is isomorphic to \mathcal{M}_C . Hence the theorem follows from Proposition 4.1. \square

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